Projectivity criterion of Moishezon spaces and density of projective symplectic varieties

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Abstract.

A Moishezon manifold is a projective manifold if and only if it is a Kähler manifold [Mo 1]. However, a singular Moishezon space is not generally projective even if it is a Kähler space [Mo 2]. Vuono [V] has given a projectivity criterion for Moishezon spaces with isolated singularities. In this paper we shall prove that a Moishezon space with 1-rational singularities is projective when it is a Kähler space (Theorem 6).

We shall use Theorem 6 to show the density of projective symplectic varieties in the Kuranishi family of a (singular) symplectic variety (Theorem 9), which is a generalization of the result by Fujiki [Fu 1, Theorem 4.8] to the singular case.

In the Appendix we give a supplement and a correction to the previous paper [Na] where singular symplectic varieties are dealt with.

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1. Projectivity of a Kähler Moishezon variety

In this paper, a compact complex variety means a compact, irreducible and reduced complex space. We mean by a Moishezon variety X a compact complex variety X with n algebraically independent meromorphic functions, where $n = \dim X$, [Mo 1].

For a complex space X we say that X is Kähler if X admits a Kähler metric (form) in the sense of [Gr], [Mo 2] (cf.[B]).

Definition 1. Let X be a compact complex variety.

(1) An element $b \in H_2(X, \mathbf{Q})$ is an *analytic homology class* if b is represented by a 2-cycle $\Sigma\alpha_j C_j$ where C_j are complex subvarieties of dimension 1 and $\alpha_j \in \mathbf{Q}$. Denote by $A_2(X, \mathbf{Q})$ the subspace of $H_2(X, \mathbf{Q})$ spanned by analytic homology classes. Define $A_2(X, \mathbf{R}) := A_2(X, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{R}$.

(2) For an element $b \in A_2(X, \mathbf{Q})$ and a line bundle L of X, the intersection number (b, L) is well-defined. Two elements b, b' of $A_2(X, \mathbf{Q})$ are said to be numerically equivalent if $(b, L) = (b', L)$ for all line bundles L on X. Denote by

$N_1(X)_{\mathbf{Q}}$ the quotient $\mathbf{Q}$-vector space of $A_2(X, \mathbf{Q})$ by this numerical equivalence. Define $N_1(X)_{\mathbf{R}} := N_1(X)_{\mathbf{Q}} \otimes \mathbf{R}$.

(3) Define $\mathrm{Pic(X)}_{\mathbf{Q}} := \mathrm{Pic(X)} \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathrm{Pic(X)}_{\mathbf{R}} := \mathrm{Pic(X)} \otimes_{\mathbf{Z}} \mathbf{R}$. Two line bundles L and L' are numerically equivalent if $(b, L) = (b, L')$ for all $b \in A_2(X, \mathbf{Q})$. Let $N^1(X)$ be the abelian group of numerical classes of line bundles on X. Define $N^1(X)_{\mathbf{Q}} := N^1(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $N^1(X)_{\mathbf{R}} := N^1(X) \otimes_{\mathbf{Z}} \mathbf{R}$.

Proposition 2([Ko-Mo, (12.1.5)]) *Let X be a Moishezon variety with 1-rational singularities, that is, X is normal and has a resolution $\pi : Y \to X$ such that $R^1\pi_*\mathcal{O}_Y = 0$. Then an analytic homology class $b \in A_2(X, \mathbf{Q})$ is zero if it is numerically equivalent to 0. In particular, $A_2(X, \mathbf{Q}) = N_1(X)_{\mathbf{Q}}$.*

Sketch of proof: Let $\pi : Y \to X$ be a resolution such that Y is projective. Let $N_1(Y/X)_{\mathbf{Q}}$ be the subspace of $N_1(Y)_{\mathbf{Q}}$ generated by the classes of curves contained in a fiber of π. We have an exact sequence

$$0 \to N_1(Y/X)_{\mathbf{Q}} \to N_1(Y)_{\mathbf{Q}} \to N_1(X)_{\mathbf{Q}} \to 0.$$

We use the condition $R^1\pi_*\mathcal{O}_Y = 0$ to prove the middle exactness of the sequence.

We may assume that b is represented by a curve C on X. By cutting out $\pi^{-1}(C)$ by general hyperplane sections of Y, we can find a curve C' such that, as a cycle, $\pi_*(C') = mC$ for a positive integer m. Define $b' := [(1/m)C'] \in N_1(Y)_{\mathbf{Q}}$. By the exact sequence, $b' \in N_1(Y/X)_{\mathbf{Q}}$. Thus b' is represented by a ($\mathbf{Q}$)curve D contained in some fibers of π. By definition, D and $(1/m)C'$ are numerically equivalent. Since numerical equivalence and homological equivalence coincide on Y (cf. [Mo 1, page 83, Theorem 9]), these are also homological equivalent. Therefore, $\pi_*(D)$ and $\pi_*((1/m)C')$ are homologically equivalent on X. Since $[\pi_*(D)] = 0$, $b = [\pi_*((1/m)C')] = 0$.

Lemma 3. *Let X be a Kähler, Moishezon variety with a Kähler form ω. Assume that an element $L \in \mathrm{Pic(X)}_{\mathbf{Q}}$ satisfies the following condition $(**)$.*

(**) *For every curve $C \subset X$,*

$$\int_C \omega = (C.L).$$

Then, for any subvariety $W \subset X$ of dimension k, $(L^k)_W > 0$.

Proof. By the assumtion, L is nef, hence $(L^k)_W \geq 0$ (cf. [Kl]). We shall derive a contradiction by assuming that $(L^k)_W = 0$.

Put $\omega' := \omega|_W$. Then ω' becomes a Kähler form of W, and W is a Kähler Moishezon variety. Let $p \in W$ be a smooth point, and let $h : \hat{W} \to W$ be the blowing up at p. Denote by E the inverse image $h^{-1}(p)$. E is a Cartier divisor of $\hat{W}$. By using a Hermitian metric on $\mathcal{O}_{\hat{W}}(E)$, one can define a d-closed $(1,1)$-form α_E on $\hat{W}$ in such a way that $\omega_\epsilon := h^*\omega - \epsilon\alpha_E$ becomes a Kähler form on $\hat{W}$ if $\epsilon > 0$ is sufficiently small.

Fix a sufficiently small rational number $\epsilon > 0$. Put $L' := L|_W$ and $F := h^*L' - \epsilon E$. By the condition $(**)$, for every curve $C \subset \hat{W}$, we have $(C.F) = \int_C (h^*\omega' - \epsilon\alpha_E)$.

In particular, F is a nef $\mathbf{Q}$-line bundle on $\hat{W}$. Therefore, $(F)^k \geq 0$.

On the other hand, since $(h^*L')^i(E)^{k-i} = 0$ for $i \neq 0, k$, we have $(F)^k = (L')^k + \epsilon^k(-E)^k$. By the assumption, $(L')^k = 0$. By the definition of E, we have $(-E)^k < 0$. Hence $(F)^k < 0$, a contradiction.

Theorem 4 (cf. [Mo 1, page 77, Theorem 6]) *Let X be a Moishezon variety. Assume that an element $L \in \mathrm{Pic(X)}_{\mathbf{Q}}$ satisfies the inequality $(L)^k_W > 0$ for every k dimensional subvariety W of X. Then L is ample.*

By Lemma 3 and Theorem 4 we have the following corollary.

Corollary 5. *Let X be a Kähler Moishezon variety with a Kähler form ω. Assume that an element $L \in \mathrm{Pic(X)}_{\mathbf{Q}}$ satisfies the equality for any curve $C \subset X$:*

$$(C.L) = \int_C \omega.$$

Then L is ample.

Theorem 6. *Let X be a Moishezon variety with 1-rational singularities (cf. Proposition 2). If X is Kähler, then X is projective.*

Proof.

Since the numerical equivalence and the homological equivalence coincide for (analytic) 1-cycle by Proposition 2, we have a natural map $\alpha : N^1(X)_{\mathbf{Q}} \to (A_2(X, \mathbf{Q}))^*$ and α is an isomorphism.

Taking the tensor product with $\mathbf{R}$, we have a map $\alpha_{\mathbf{R}} : N^1(X)_{\mathbf{R}} \to (A_2(X, \mathbf{R}))^*$ and $\alpha_{\mathbf{R}}$ is an isomorphism. By the 2-nd cohomolgy class defined by the Kähler form ω (cf. [B, (4.15)]) one can regard the Kähler form as an element of $(A_2(X, \mathbf{R}))^*$. Since $\alpha_{\mathbf{R}}$ is surjective, there is an element $d \in N^1(X)_{\mathbf{R}}$ such that $(C.d) = \int_C \omega$ for every curve $C \subset X$.

Approximate $d \in N^1(X)_{\mathbf{R}}$ by a convergent sequence $\{d_m\}$ of rational elements $d_m \in N^1(X)_{\mathbf{Q}}$.

Let us fix the basis $b_1, ..., b_l$ of the vector space $N^1(X)_{\mathbf{Q}}$. Each b_i is represented by an element $B_i \in \mathrm{Pic(X)}$. Now d (resp. d_m) is represented by an element in $\mathrm{Pic(X)}_{\mathbf{R}}$ (resp. $\mathrm{Pic(X)}_{\mathbf{Q}}$) $D := \Sigma x_i B_i$ (resp. $D_m := \Sigma x_i^{(m)} B_i$) such that $\lim x_i^{(m)} = x_i$.

Put $E_m := D_m - D$. Then there are d closed $(1,1)$-forms α_m corresponding to E_m such that $\{\alpha_m\}$ uniformly converge to 0.

If m is chosen sufficiently large, then $\omega_m := \omega + \alpha_m$ is a Kähler form. Since

$$(C.D_m) = \int_C \omega_m$$

for every curve $C \subset X$, we see that D_m is ample by Corollary 5.

Corollary 6'. *Let X be a Moishezon variety with rational singularities. If X is Kähler, then X is projective.*

Remark. If we do not assume that X has 1-rational singularities, Theorem 6 is no longer true (cf. [Mo 2]).

2. Application: Density of projective symplectic varieties

A symplectic variety is a compact normal Kähler space X with the following properties: (1) The regular part U of X has an everywhere non-degenerate holomorphic 2-form Ω, and (2) for a (any) resolution of singularities $f : \tilde{X} \to X$ such that $f^{-1}(U) \cong U$, the 2-form Ω extends to a holomorphic 2-form on $\tilde{X}$. Here the extended 2-form may possibly degenerate along the exceptional locus. By definition, X has only canonical singularities, hence has only rational singularities.

If X has a resolution $f : \tilde{X} \to X$ such that Ω extends to an everywhere non-degenerate 2-form on $\tilde{X}$, then we say that X has a symplectic resolution.

Symplectic varieties with no symplectic resolutions are constructed as symplectic V-manifolds in [Fu 1]. Recently, O'Grady [O] has constructed such varieties as the moduli spaces of semi-stable torsion free sheaves on a polarized K3 surface (cf. [Na, Introduction]). His examples are no more V-manifolds.

These examples satisfy the following condition:

(*): The natural restriction map

$$H^2(X, \mathbf{Q}) \cong H^2(U, \mathbf{Q})$$

is an isomorphism.

In [Na] we have formulated the local Torelli problem for these symplectic varieties, and proved it. More precisely, we have proved it for a symplectic variety X with the following properties.

(a): $\mathrm{Codim}(\Sigma \subset \mathrm{X}) \geq 4$, where $\Sigma := \mathrm{Sing(X)}$,

(b): $h^1(X, \mathcal{O}_X) = 0$, $h^0(U, \Omega^2_U) = 1$, and

(c): (*) is satisfied.[1]

Let X be a symplectic variety satisfying (a), (b) and (c). Let $0 \in S$ be the Kuranishi space of X and $\bar{\pi} : \mathcal{X} \to S$ be the universal family such that $\bar{\pi}^{-1}(0) = X$. Let $\mathcal{U}$ be the locus in $\mathcal{X}$ where $\bar{\pi}$ is a smooth map. We denote by π the restriction $\bar{\pi}$ to $\mathcal{U}$. S is nonsingular by the condition (a). Note that every fiber of $\bar{\pi}$ is a symplectic variety satisfying (a), (b) and (c) (cf. [Na]).

The cohomology $H^2(U, \mathbf{C})$ admits a natural mixed Hodge structure because U is a Zariski open subset of a compact Kähler space (cf. [Fu 2]). In our case,

[1] Note that this condition is equivalent to the condition $(*)$ in [Na, Remark (2)] (cf. (b) in the proof of [Na, Prop. 9]).

because of condition (a), it is pure of weight 2, and the Hodge decomposition is given by

$$H^2(U, \mathbf{C}) = H^0(U, \Omega_U^2) \oplus H^1(U, \Omega_U^1) \oplus H^2(U, \mathcal{O}_U).$$

For details of this, see the footnote of the e-print version of [Na, p.21]. Moreover, the tangent space $T_{S,0}$ at the origin is canonically isomorphic to $H^1(U, \Theta_U)$. By a holomorphic symplectic 2-form Ω, $H^1(U, \Theta_U)$ is identified with $H^1(U, \Omega_U^1)$.

Let us fix a resolution $\tilde{X}$ of X. For $\alpha \in H^2(U, \mathbf{C})$, denote by $\tilde{\alpha} \in H^2(\tilde{X}, \mathbf{C})$ the image of α by the map $H^2(U, \mathbf{C}) \cong H^2(X, \mathbf{C}) \to H^2(\tilde{X}, \mathbf{C})$. The holomorphic symplectic 2-form Ω defines a holomorphic 2-form on $\tilde{X}$ by the definition of a symplectic variety. We denote by the same symbol Ω this holomorphic 2-form on $\tilde{X}$. Here we normalize Ω so that $\int_{\tilde{X}} (\Omega\overline{\Omega})^l = 1$.

One can define a quadratic form q on $H^2(U, \mathbf{C})$ by

$$q(\alpha) := l/2 \int_{\tilde{X}} (\Omega\overline{\Omega})^{l-1} \tilde{\alpha}^2 + (1-l) \int_{\tilde{X}} \Omega^l \overline{\Omega}^{l-1} \tilde{\alpha} \int_{\tilde{X}} \Omega^{l-1} \overline{\Omega}^l \tilde{\alpha},$$

where $\dim X = 2l$. Note that q is independent of the choice of the resolution $\tilde{X}$. The quadratic form q is defined over $H^2(U, \mathbf{R})$. By the same argument as [Be, Théorème 5, (a), (c)], we can write $q(\alpha) = c_1(\int_{\tilde{X}} [f^*\omega]^{2l-2} \tilde{\alpha}^2) - c_2(\int_{\tilde{X}} [f^*\omega]^{2l-1} \tilde{\alpha})^2$ by a Kähler form ω on X and by suitable positive real constants c_1 and c_2. Let us define $H_0^2(U, \mathbf{R}) := \{\alpha \in H^2(U, \mathbf{R}); \int_{\tilde{X}} [f^*\omega]^{2l-1} \tilde{\alpha} = 0\}$. Then we have a direct sum decomposition $H^2(U, \mathbf{R}) = H_0^2(U, \mathbf{R}) \oplus \mathbf{R}[\omega]$, where $H_0^2(U, \mathbf{R})$ and $\mathbf{R}[\omega]$ are orthogonal with respect to q.

We shall prove that $q([\omega]) > 0$.

It is easily checked that $q([\omega]) = l/2 \int_{\tilde{X}} (\Omega\overline{\Omega})^{l-1} [f^*\omega]^2$. We assume that the resolution $f : \tilde{X} \to X$ is obtained by a succession of blowing ups with smooth centers contained in the singular locus. Let $\{E_i\}$ be the exceptional divisors of f. Then, for sufficiently small positive real numbers ϵ_i, $[f^*\omega] - \Sigma\epsilon_i[E_i]$ is a Kähler class on $\tilde{X}$ (cf. [Fu 3, Lemma 2]). By [W, Corollaire au Théorème 7, p.77] we have $l/2 \int_{\tilde{X}} (\Omega\overline{\Omega})^{l-1} ([f^*\omega] - \Sigma\epsilon_i[E_i])^2 > 0$. On the other hand, by [Na, Remark (1), p. 24] we have $\int_{\tilde{X}} (\Omega\overline{\Omega})^{l-1} [f^*\omega][E_i] = \int_{\tilde{X}} (\Omega\overline{\Omega})^{l-1} [E_i]^2 = 0$; hence we have $q([\omega]) > 0$.

Denote by $Q : H^2(U, \mathbf{C}) \times H^2(U, \mathbf{C}) \to \mathbf{C}$ the symmetric bilinear form defined by q. With respect to Q, $H^0(U, \Omega_U^2) \oplus H^2(U, \mathcal{O}_U)$ is orthogonal to $H^1(U, \Omega_U^1)$. Let us define $N := \{v \in H^1(U, \Omega_U^1); Q(v, x) = 0 \text{ for any } x \in H^1(U, \Omega_U^1)\}$. Since $q([\omega]) > 0$ for a Kähler form ω on X, N does not coincide with $H^1(U, \Omega_U^1)$. By the identification of $H^1(U, \Omega_U^1)$ with $T_{S,0}$, we regard N as a subspace of $T_{S,0}$.

Later we shall prove that the quadratic form q is non-degenerate and, in fact, $N = 0$. But, before doing this, we first prove the density of projective symplectic varieties in a rather imcomplete form (cf. Proposition 7 below). After that we will show that q is non-degenerate by using Proposition 7. As a consequence,

we will see that, in Proposition 7, the assumption for $T_{S_1,0}$ is not necessary; hence we can prove the density in a complete form (cf. Theorem 9).

Proposition 7. *Notation and assumptions being the same as above, let $0 \in S_1 \subset S$ be a positive dimensional non-singular subvariety of S such that $T_{S_1,0}$ is not contained in N. Then, for any open neighborhood $0 \in V \subset S$, there is a point $s \in V \cap S_1$ such that $\mathcal{X}_s$ is a projective symplectic variety.*

Proof. We may assume that $\dim S_1 = 1$. Denote by $\mathcal{X}_1$ the fiber product $\mathcal{X} \times_S S_1$ and denote by $\bar{\pi}_1$ the induced map from $\mathcal{X}_1$ to S_1. Take a resolution of singularities $\nu : \mathcal{Y}_1 \to \mathcal{X}_1$ in such a way that ν is an isomorphism over smooth locus of $\mathcal{X}_1$. We also assume that ν is obtained by the succession of blowing ups with smooth centers. So there exists a ν-ample divisor of the form $-\Sigma\epsilon_i\mathcal{E}_i$, where $\mathcal{E}_i$ are ν-exceptional divisors and ϵ_i are positive rational numbers.

Let S_1^0 be the set of points $s \in S_1$ where $(\bar{\pi}_1 \circ \nu)^{-1}(s)$ are smooth. Then S_1^0 is a non-empty Zariski open subset of S_1.

We may assume that $0 \in S_1^0$. In fact, if $0 \notin S_1^0$, then we take a point $s \in S_1^0 \cap V$. Then the family $\bar{\pi} : \mathcal{X} \to S$ can be regarded as the Kuranishi family of $\mathcal{X}_s$ near $s \in S$ because $H^0(X, \Theta_X) = 0$ by the condition (b). For this point $s \in S$, $\mathcal{X}_s$ satisfies all conditions (a), (b) and (c). So, if the theorem holds for $\mathcal{X}_s$, then we can find a point $s' \in S_1^0 \cap V$ where $\mathcal{X}_{s'}$ is projective.

In the remainder we shall assume that $0 \in S_1^0$. Thus, if S is chosen sufficiently small, then $\nu : \mathcal{Y}_1 \to \mathcal{X}_1$ is a simultaneous resolution of $\{\mathcal{X}_{1,s}\}$, $s \in S_1$.

Claim *For any open neighborhood $0 \in V \subset S$, there is a point $s \in V \cap S_1$ such that $\mathcal{Y}_{1,s}$ is a projective variety.*

If the claim is justified, then, for such a point s, $\mathcal{X}_{1,s}$ is a Moishezon variety. On the other hand, $\mathcal{X}_{1,s}$ is a symplectic variety satisfying (a), (b) and (c) (cf. [Na]). In particular, $\mathcal{X}_{1,s}$ is a Kähler Moishezon variety with rational singularities. By Theorem 6, we conclude that $\mathcal{X}_{1,s}$ is projective.

Proof of Claim.

(i): Put $Y = \mathcal{Y}_{1,0}$. By definition of $\mathcal{X}_1$, $\mathcal{X}_{1,0} = X$. The bimeromorphic map $\nu_0 : Y \to X$ is a resolution of singularities. Put $E_i := \mathcal{E}_{i,0}$ where $\mathcal{E}_i$ are ν-exceptional divisors. By the construction of ν, there are positive rational numbers ϵ_i such that $-\Sigma\epsilon_i E_i$ is ν_0-ample.

(ii): We have a constant sheaf $R^2\bar{\pi}_*\mathbf{C}$ on S. There is an isomorphism $R^2\bar{\pi}_*\mathbf{C} \otimes_{\mathbf{C}} \mathcal{O}_S \cong R^2\pi_*\mathbf{C} \otimes_{\mathbf{C}} \mathcal{O}_S$. The right hand side is filtered as $R^2\pi_*\mathbf{C} \otimes_{\mathbf{C}} \mathcal{O}_S = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \mathcal{F}^2 \supset 0$ in such a way that $Gr^i_{\mathcal{F}} = R^{2-i}\pi_*\Omega^i_{\mathcal{U}/S}$.

Over each point $s \in S$, this filtration gives the Hodge decomposition of $H^2(\mathcal{X}_s)$. The natural mixed Hodge structure on $H^2(\mathcal{X}_s)$ is pure, and its (i,j)-component $H^{i,j}(\mathcal{X}_s)$ $(i+j=2)$ is given by $H^j(\mathcal{U}_s, \Omega^i_{\mathcal{U}_s})$.

For $a \in H^2(X, \mathbf{R}) = \Gamma(S, R^2\bar{\pi}_*\mathbf{R})$, define S_a to be the locus in S where $a \in H^{1,1}(\mathcal{X}_s)$.

Let ω be a Kähler form on X and denote by $[\omega] \in H^2(X, \mathbf{R}) (= H^2(U, \mathbf{R}))$ its

cohomology class (cf. [B, (4.15)]). Then the tangent space $T_{S_{[\omega]},0}$ is isomorphic to $\{v \in H^1(U, \Omega^1_U); Q(v, [\omega]) = 0\}$, where we identify $T_{S,0}$ with $H^1(U, \Omega^1_U)$ as explained above. Let $v_1 \in H^1(U, \Omega^1_U)$ be a generator of 1-dimensional vector space $T_{S_1,0}$.

Define a linear map $Q_{v_1} : H^1(U, \Omega^1_U) \to \mathbf{C}$ by $Q_{v_1}(x) = Q(v_1, x)$. By the assumption, this map is surjective. By definition, $\mathrm{Ker(Q_{v_1})}$ is a complex hyperplane in $H^1(U, \Omega^1_U)$. Therefore, $H^1(U, \Omega^1_U) \cap H^2(U, \mathbf{R})$ is not contained in $\mathrm{Ker(Q_{v_1})}$. Hence we can find an element a in any small open neighborhood of $[\omega] \in H^{1,1}(X) \cap H^2(X, \mathbf{R})$ in such a way that S_a intersects S_1 in 0 transversely.

(iii): Let α_{E_i} be a d-closed $(1,1)$ form on Y corresponding to E_i. If necessary, by multiplying to $\{\epsilon_i\}$ a sufficiently small positive rational number simultaneously, the d-closed (1,1) form

$$(\nu_0)^*\omega - \Sigma\epsilon_i\alpha_{E_i}$$

becomes a Kähler form on Y (cf. [Fu 3, Lemma 2]).

Take $a \in H^{1,1}(X) \cap H^2(X, \mathbf{R})$ as in the final part of (ii). Then we can find a suitable d-closed $(1,1)$-form ω' on Y which represents $(\nu_0)^*a \in H^{1,1}(Y) \cap H^2(Y, \mathbf{R})$ in such a way that

$$\gamma := \omega' - \Sigma\epsilon_i\alpha_{E_i}$$

becomes a Kähler form on Y.

$[E_i]$ remains of $(1,1)$ type in $H^2(\mathcal{Y}_{1,s})$ for arbitrary $s \in S_1$ because the Cartier divisor E_i extends sideways in the family $\mathcal{Y}_1 \to S_1$. Hence, by the choice of a, $(\nu_0)^*a - \Sigma\epsilon_i[E_i]$ is no more of type $(1,1)$ for $s \neq 0$ sufficiently near 0.

(iv): The last step is the same as the proof of [Fu 1, Theorem 4.8, (2)]. Approximate $a \in H^{1,1}(X) \cap H^2(X, \mathbf{R})$ by a sequence of rational classes $\{a_m\}$ ($a_m \in H^2(X, \mathbf{Q})$).

We put $b := (\nu_0)^*a - \Sigma\epsilon_i[E_i]$ and $b_m := (\nu_0)^*a_m - \Sigma\epsilon_i[E_i]$.

We take a C^∞ family of Kähler forms $\{\gamma_s\}$ on $\{\mathcal{Y}_{1,s}\}$ such that $\gamma_0 = \gamma$. For each $s \in S_1$, the cohomolgy class b_m is represented by a unique harmonic 2-form $(\omega_m)(s)$ on $\mathcal{Y}_{1,s}$ with respect to γ_s. If m is large, then $(\omega_m)(s)$ becomes a Kähler form on $\mathcal{Y}_{1,s}$ for some point $s \in V \cap S_1$. Since b_m is a rational cohomolgy class, $\mathcal{Y}_{1,s}$ is a projective manifold by [Ko] for this s.

Corollary 8. *Let X be a symplectic variety of dim $n = 2l$ satisfying (a), (b) and (c). Let q be the quadratic form on $H^2(U, \mathbf{R})$ defined above. Then q is non-degenerate and has signature $(3, B-3)$ where $B := \dim H^2(U, \mathbf{R})$.*

Proof. We first prove this collorary when X is projective. Let ω be a Kähler form that comes from a very ample line bundle L on X. Take general global sections t_1, ..., t_{n-2} of L. We put $T_0 = X$ and, for $1 \leq i \leq n-2$,

define $T_i \subset X$ to be the common zeros of t_1, ..., t_i. Put $\Sigma_i = \Sigma \cap T_i$, where $\Sigma := \mathrm{Sing(X)}$. By the condition (a), $\Sigma_{n-2} = \emptyset$ and T_{n-2} is a nonsingular surface. For simplicity we put $T := T_{n-2}$. By [H, Theorem 2], the pair $(T_i - \Sigma_i, T_{i+1} - \Sigma_{i+1})$ is $n-i-1$-connected. In particular, the restriction maps $H^2(T_i \setminus \Sigma_i, \mathbf{R}) \to H^2(T_{i+1} \setminus \Sigma_{i+1}, \mathbf{R})$ are all injective. Hence $H^2(U, \mathbf{R}) \to H^2(T, \mathbf{R})$ is an injection. Note that both sides have pure Hodge structures of weight 2 (cf. the footnote on p.21 of the e-print version of [Na]) and this injection is a morphism of Hodge structures. Define $H^2_0(T, \mathbf{R}) := \{\alpha \in H^2(T, \mathbf{R}); (\omega|_T).\alpha = 0\}$. Then we have the restriction map $H^2_0(U, \mathbf{R}) \to H^2_0(T, \mathbf{R})$, which is an injection. This restriction map induces an injection $H^{1,1}_0(U)_{\mathbf{R}} \to H^{1,1}_0(T)_{\mathbf{R}}$, where $H^{1,1}_0(U)_{\mathbf{R}} := H^{1,1}(U) \cap H^2_0(U, \mathbf{R})$ and $H^{1,1}_0(T)_{\mathbf{R}} := H^{1,1}(T) \cap H^2_0(T, \mathbf{R})$. Let q' be the quadratic form on $H^{1,1}_0(T)_{\mathbf{R}}$ defined by the cup product. By the definition of $H^2_0(U, \mathbf{R})$, $q|_{H^{1,1}_0(U)_{\mathbf{R}}} = c_1 q'|_{H^{1,1}_0(U)_{\mathbf{R}}}$ for a suitable positive real constant c_1. Since $q'|_{H^{1,1}_0(U)_{\mathbf{R}}}$ is negative-definite, $q|_{H^{1,1}_0(U)_{\mathbf{R}}}$ is also negative-definite. We have a direct sum decomposition with respect to q:

$$H^2(U, \mathbf{R}) = \mathbf{R}[\omega] \oplus (H^{2,0}(U) \oplus H^{0,2}(U))_{\mathbf{R}} \oplus H^{1,1}_0(U)_{\mathbf{R}},$$

where $(H^{2,0}(U) \oplus H^{0,2}(U))_{\mathbf{R}} := (H^{2,0}(U) \oplus H^{0,2}(U)) \cap H^2(U, \mathbf{R})$.

Since q is positive-definite on the first two factors, q has signature $(3, B-3)$. Therefore Corollary 8 has been proved when X is projective.

When X is non-projective, by Proposition 7, we can find a point s in any small open neighborhood V of $0 \in S := \mathrm{Def(X)}$ in such a way that $\mathcal{X}_s$ is a projective symplectic variety. Since, for each $s \in S$, $\mathcal{X}_s$ is a symplectic variety with $h^0(\mathcal{U}_s, \Omega^2_{\mathcal{U}_s}) = 1$, we can find a symplectic form Ω_s on each $\mathcal{X}_s$ so that $\int_{\mathcal{U}_s}(\Omega_s \overline{\Omega}_s)^l = 1$. Let q_s be the quadratic form on $H^2(\mathcal{U}_s, \mathbf{R})$ defined by Ω_s. By using a natural flat structure in $R^2\pi_*\mathbf{C} \otimes_{\mathbf{C}} \mathcal{O}_S$ ([Na, Theorem 8,(1)]), $H^2(U, \mathbf{R})$ and $H^2(\mathcal{U}_s, \mathbf{R})$ are identified. By the same argument as [Be, Théorème 5], we see that q and q_s are proportional (by a positive constant) under this identification. Because $\mathcal{X}_s$ is projective, q_s has signature $(3, B-3)$. Hence q also has signature $(3, B-3)$.

Theorem 9. *Notation and assumptions being the same as above, let* $0 \in S_1 \subset S$ *be a positive dimensional non-singular subvariety Then, for any open neighborhood* $0 \in V \subset S$*, there is a point* $s \in V \cap S_1$ *such that* $\mathcal{X}_s$ *is a projective symplectic variety.*

Proof. It suffices to prove that $N = 0$ in Proposition 7. But this follows from Corollary 8.

Appendix: Supplement to [Na]

In this appendix, we shall claim that Theorem 4 of [Na] remains true under a weaker condition. The exact statement is the following.

Theorem A-4. *Let X be a Stein open subset of a complex algebraic variety. Assume that X has only rational singularities. Let Σ be the singular locus of X and let $f : Y \to X$ be a resolution of singularities such that $f|_{Y \setminus f^{-1}(\Sigma)} : Y \setminus f^{-1}(\Sigma) \cong X \setminus \Sigma$. Then $f_*\Omega^2_Y \cong i_*\Omega^2_U$ where $U := X \setminus \Sigma$ and $i : U \to X$ is a natural injection.*

Theorem 4 in [Na] was stated under the condition that X has rational Gorenstein singularities. We shall roughly sketch how to modify the original proof to drop the Gorenstein condition.

The first step is to drop the Gorenstein condition from Proposition 1 of [Na]:

Proposition A-1. *Let X be a Stein open subset of a complex algebraic variety. Assume that X has only rational singularities. Let Σ be the singular locus of X and let $f : Y \to X$ be a resolution of singularities such that $f|_{Y \setminus f^{-1}(\Sigma)} : Y \setminus f^{-1}(\Sigma) \cong X \setminus \Sigma$ and $D := f^{-1}(\Sigma)$ is a simple normal crossing divisor. Then $f_*\Omega^2_Y(\log D) \cong i_*\Omega^2_U$ where $U := X \setminus \Sigma$ and $i : U \to X$ is a natural injection.*

To prove Proposition 1, we have first taken an element ω from $H^0(U, \Omega^2_U)$, and have shown that ω has at worst a log pole at each irreducible component F of D. When X has rational Gorenstein singularities, $(X, p) \cong (R.D.P) \times (\mathbf{C}^{n-2}, 0)$ for all singular points $p \in X$ outside certain codimension 3 (in X) locus $\Sigma_0 \subset \Sigma$. Since the proposition holds around such points, we only had to consider the case $f(F) \subset \Sigma_0$.

In our general case, we have to take all irreducible components F of D into consideration. So we put $k := \dim \Sigma - \dim f(F)$, $l := \mathrm{Codim}(\Sigma \subset \mathrm{X})$ and continue the same argument as [Na, Proposition 1]. Here we note that a general hyperplane section H of X has again rational singularities. In proving Claim of (a-2), we have used the following vanishing (the notation being the same as [Na]):

$R^i\pi_*\Omega^{l-2}_{Y_t}(\log D_t)(-D_t) = R^i\pi_*\Omega^{l-1}_{Y_t}(\log D_t)(-D_t) = R^i\pi_*\omega_{Y_t} = 0$ for $i \geq l-1$ and for $t \in \Delta$.

Except the following cases, these vanishings follow from [St]:

$l = 3$: $R^2\pi_*\Omega^1_{Y_t}(\log D_t)(-D_t)$,

$l = 2$: $R^1\pi_*\Omega^1_{Y_t}(\log D_t)(-D_t)$, $R^2\pi_*\mathcal{O}_{Y_t}(-D_t)$, $R^1\pi_*\mathcal{O}_{Y_t}(-D_t)$.

For theses exceptional cases, we can prove the vanishing by combining the method in the proof of [Na-St, Theorem (1.1)] and the fact that a rational singularity is Du Bois [Kov].

In proving Claim of (b-2) we also need similar vanishings; but they are already contained in the above cases.

Next we shall generalize Lemma 2 of [Na] as follows:

Lemma A-2. *Let $p \in X$ be a Stein open neighborhood of a point p of a complex algebraic variety. Assume that X is a rational singularity of $\dim X \geq 2$.*

Let $f : Y \to X$ be a resolution of singularities of X such that $E := f^{-1}(p)$ is a simple normal crossing divisor. Then $H^0(Y, \Omega^i_Y) \to H^0(Y, \Omega^i_Y(\log E))$ are isomorphisms for $i = 1, 2$.

Lemma 2 of [Na] was stated under the condition $\dim X \geq 3$. In the proof of [Na, Lemma 2] we have first shown that $H^3_E(Y, \mathbf{C}) \to H^3(Y, \mathbf{C})$ and $H^2_E(Y, \mathbf{C}) \to H^2(Y, \mathbf{C})$ are both injective. Even when $\dim X = 2$, these are true. First note that when $\dim X = 2$, (X, p) is an isolated singularity. By taking the dual of the first map, we get the map $H^1_E(Y, \mathbf{C}) \to H^1(E, \mathbf{C})$. Since X is a rational singularity, $H^1(E, \mathbf{C}) = 0$; thus the dual map is surjective.

The injectivity of the second map follows from the next observation: $H^2(Y, \mathbf{C}) \cong H^1(Y, \mathcal{O}^*_Y) \otimes \mathbf{C}$, and $H^2_E(Y, \mathbf{C}) \cong \oplus \mathbf{C}[E_i]$ where E_i are irreducible components of E.

The rest of the proof of Lemma A-2 is the same as [Na, Lemma 2].

The following remark is quite similar to [Na, Remark below Lemma 2]:

Remark A. *In Lemma A-2, the map*

$$H^0(E, \Omega^i_Y / \Omega^i_Y(\log E)(-E)) \to H^0(E, \Omega^i_Y(\log E)/\Omega^i_Y(\log E)(-E))$$

is surjective for $i = 1, 2$

Finally we shall prove

Proposition A-3. *Let X be a Stein open subset of a complex algebraic variety. Assume that X has only rational singularities. Let Σ be the singular locus of X and let $f : Y \to X$ be a resolution of singularities such that $D := f^{-1}(\Sigma)$ is a simple normal crossing divisor and such that $f|_{Y \setminus D} : Y \setminus D \cong X \setminus \Sigma$. Then $f_*\Omega^2_Y \cong f_*\Omega^2_Y(\log D)$.*

This is a generalization of Proposition 3 of [Na], in which the same result was stated under the condition that X has rational Gorenstein singularities.

To prove Proposition 3, we have first taken an element ω from $H^0(X, f_*\Omega^2_Y(\log D))$, and have shown that ω is regular along each irreducible component F of D. When X has rational Gorenstein singularities, $(X, p) \cong (R.D.P) \times (\mathbf{C}^{n-2}, 0)$ for all singular points $p \in X$ outside certain codimension 3 (in X) locus $\Sigma_0 \subset \Sigma$. Since the proposition holds around such points, we only had to consider the case $f(F) \subset \Sigma_0$.

In our general case, we have to take all irreducible components F of D into consideration. So we put $k := \dim \Sigma - \dim f(F)$, $l := \mathrm{Codim}(\Sigma \subset \mathrm{X})$ and continue the same argument as [Na, Proposition 3]. The rest of the argument is similar to [Na, Proposition 3]. In the original proof we have used [Na, Remark below Lemma 2], but now we shall use Remark A.

Correction to [Na]. In the introduction of [Na], a conjecture has been posed as a generalization of Bogomolov splitting theorem. This conjecture should be :

Let Y be a smooth projective variety over $\mathbf{C}$ with Kodaira dimension 0. Then there is a finite cover $\pi : Y' \to Y$ such that (a) π is etale outside the support of the pluri-canonical divisor of Y, and (b) Y' is birationally equivalent to $Y_1 \times Y_2 \times Y_3$, where Y_1 is an Abelian variety, Y_2 is a symplectic variety, and Y_3 is a Calabi-Yau variety.

In the old version, π was assumed to be a finite etale cover.

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